Why Closed, Bounded Sets in $\mathbb{R}^n$ are Compact

Suppose A is a closed, bounded subset of $\mathbb{R}^n$. Then $\exists$ M>0 such that $A \subset \{(x_1,\ldots x_n) \in \mathbb{R}^n : |x_j| \leq M, \forall\ j\} = B$. That A is compact will follow from combining two observations:

i. a closed subset of a compact set is compact
ii. the set B is compact

To prove (i) : Suppose $A_1 \subset A_2$ with A_2 compact and A_1 a closed subset of $\mathbb{R}^n$. If $\{U_\lambda\}$, $\lambda \in \Lambda$, is a covering of A_1 (i.e., $A_1 \subset \bigcup_{\lambda \in \Lambda} U_\lambda$) with the U_λ's open, then $\{U_\lambda\}$ together with $\mathbb{R}^n$ - A_1 is a covering of A_2 by open sets (since $\mathbb{R}^n - A$ is open). Since A_2 is compact, some finite set of these, say , $U_{\lambda 1}, \ldots U_{\lambda k}$ and (perhaps) $\mathbb{R}^2$-A_1 cover A_2. Then $U_{\lambda 1}, \ldots U_{\lambda k}$ must cover A_1 since $(\mathbb{R}^n\text{-}A_1) \cap A_1 = \varnothing$ [$\mathbb{R}^n$-A_1 does not 'help' to cover A_1 !]. Hence every open cover of A_1 has a finite subcover and A_1 is compact. □

Point (ii), that B is compact, can be proved by noting that the subdivision proof that [0,1] is covering compact has an obvious generalization to B. One replaces the divisions of [0,1] into pairs of successively smaller intervals, each half the size of the previous interval, by subdivisions of B into smaller cubes. In these, each of the x_j's is controlled to lie in a half sized interval. E.g., the first step of the argument is to observe if $\{U_\lambda\}$ is a cover of b with no finite subcover then $\exists$ a cube with all sides of length ½ $\subset$B which is not covered by finitely many U_λ. Namely, some cube of the form $\{(x_1,\ldots,x_n) : \alpha_j/2 \leq x_j \leq (\alpha_j+1)/2\}$ where each α_j is either 0 or 1 has this property. The proof continues in a way entirely analogous to the [0,1] case.